数理解析研究所講究録
1149 巻 2000 年 60-75

MOD p COHOMOLOGY ALGEBRAS OF FINITE GROUPS WITH EXTRASPECIAL SYLOW p-SUBGROUPS

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1. INTRODUCTION

Let p be a prime greater than three. In this paper we consider cohomology algebras of finite groups with extraspecial Sylow p-subgroup

$$P = \langle\, a, b \mid a^p = b^p = [a,b]^p = 1, [[a,b],a] = [[a,b],b] = 1 \,\rangle$$

of order p^3 and exponent p with coefficients in fields of characteristic p.

Integral cohomology rings of these finite groups have been investigated by some people. Among them we should mention D. J. Green [6] and Tezuka-Yagita [14]. Green's work would be the first one dealing with such finite groups and contains a useful proposition that can be applied to modular case. Tezuka and Yagita's work is a comprehensive one considering finite simple groups with P as Sylow p-subgroups and gave universally stable classes. Some of these results and methods are valid for modular cases. The present work is partly inspired by their works.

We should also mention Milgram-Tezuka [9]. There they calculated the mod 3 cohomology algebra of the Mathiew group M_{12}, whose Sylow 3-subgroup is extraspecial of order 27 and exponent 3; and they showed that the cohomology algebra is isomorphic with that of the general linear group $\mathrm{GL}(3, \mathbf{F}_3)$. They used the theory of geometry of subgroups, as the title suggests.

However, our purpose is to understand mod p cohomology algebras from a view point of modular representation theory of finite groups. Our main tools include the theory of relative projectivity of modules and theory of cohomology varieties of modules.

In Okuyama-Sasaki [11] we studied some applications of theory of relative projectivity of modules to the cohomology theory of finite groups; and we calculated the mod 2 cohomology algebras of finite groups with wreathed Sylow 2-subgroups. The crucial was to analyze a Carlson module. To do that we used Green correspondence and the theory of projectivity of modules relative to modules. In this report we apply our theory to finite groups with extraspecial Sylow p-subgroups for a prime $p > 3$; as an example we shall calculate the mod p cohomology algebra of the general linear group $\mathrm{GL}(3, \mathbf{F}_p)$. At that time of the symposium, the auther had not completed the calculation. Now, he believes that it is completed. The details is in Sasaki [12].

Mod p cohomology algebras of other finite groups in question will be investigated in another paper.

Here we fix some notation. Let k be a field. Let G be a finite group. All kG-modules are finitely generated. Let H be a subgroup of G. For a class ζ in $H^*(G,k)$ we shall sometimes write ζ_H or $\zeta_{|H}$ for the restriction $\operatorname{res}_H \zeta$. For a class η in $H^*(H,k)$ we shall write $\operatorname{tr}^G \eta$ for the corestriction $\operatorname{cor}^G \eta$. For a homogeneous element η in $H^n(H,k)$, where the degree n is even, we shall denote by $\operatorname{norm}^G \eta$ the image of Evens' norm map norm : $H^n(H,k) \longrightarrow H^{|G:H|n}(G,k)$. For g an element in G we denote by η^g the conjugate $\operatorname{con}^g \eta$ in $H^*(H^g,k)$. For kG-modules U and V we shall write $(U,V)_G$ for the space of the kG-homomorphisms $\operatorname{Hom}_{kG}(U,V)$.

2. Relative projectivity

In this section we state some results concerning with relative projectivity of modules and cohomology theory. Let p be an arbitrary prime and let k be a field of characteristic p. Let G be a finite group of order divisible by the prime p.

2.1. **Relative projectivity.** The following theorem deals with Green correspondence of indecomposable direct summands of Carlson modules .

Theorem 2.1. *Let ρ in $H^n(G,k)$ be a homogeneous element. Let U be an indecomposable direct summand of the Carlson module L_ρ of ρ with vertex D. Let H be a subgroup of G containing the normalizer $N_G(D)$ and let V be a Green correspondent of U with respect to (G,D,H). Then the Green correspondent V is a direct summand of the Carlson module $L_{(\rho_H)}$ of the restriction $\rho_H = \operatorname{res}_H \rho$ of the element ρ to the subgroup H; moreover the multiplicity of the direct summand V in $L_{(\rho_H)}$ is the same as the multiplicity of U in L_ρ.*

Next let us state briefly the theory of projectivity of modules relative to modules. Refer Okuyama-Sasaki [11] or Carlson [3] in detail.

Definition 2.1. For V a kG-module let

$$\mathcal{P}(V) = \{\, X \mid X \bigm| V \otimes A \ \exists\, A \,\}.$$

A kG-module belonging to $\mathcal{P}(V)$ above is said to be projective relative to $\mathcal{P}(V)$ or $\mathcal{P}(V)$-projective.

Definition 2.2. Let M be a kG-module. A short exact sequence $E : 0 \longrightarrow X \longrightarrow R \longrightarrow M \longrightarrow 0$ is called a $\mathcal{P}(V)$-projective cover of M if

(1) R is $\mathcal{P}(V)$-projective;
(2) the tensor product

$$0 \longrightarrow X \otimes V \longrightarrow R \otimes V \longrightarrow M \otimes V \longrightarrow 0$$

splits;
(3) the kernel X has no $\mathcal{P}(V)$-projective direct summand.

A $\mathcal{P}(V)$-projective cover of any kG-module exists and is uniquely determined up to isomorphism of sequences. Dually we can define $\mathcal{P}(V)$-injective hulls of modules.

A connection between the notion of relative projectivity above and cohomology theory is given by the following fact, which is originally due to Carlson. This will be used in Section 5. Note, however, that this is not true for $p = 2$.

Lemma 2.2. *Let p be an odd prime. Let ζ in $H^{2n}(G,k)$ be an arbitrary class. Then the extension*

$$E_\zeta : 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\zeta) \longrightarrow \Omega^{2n-1}(k) \longrightarrow 0$$

associated with ζ is a $\mathcal{P}(L_\zeta)$-projective cover of the syzygy $\Omega^{2n-1}(k)$ or equivalently a $\mathcal{P}(L_\zeta)$-injective hull of the trivial module k.

2.2. **System of parameters.** Let G have p-rank r. For $i = 1, \ldots, r$ let

$$\mathcal{H}_i(G) = \{\, C_G(E) \mid E \text{ is elementary abelian } p\text{-subgroup of rank } i \,\}.$$

Our starting point of this work is the following facts.

Theorem 2.3 (Carlson [2] Proposition 2.4). *The cohomology algebra $H^*(G,k)$ has a homogeneous system $\{\zeta_1, \ldots, \zeta_r\}$ of parameters with the property that for every $i = 1, \ldots, r$*

$$\zeta_i \in \sum_{H \in \mathcal{H}_i(G)} \operatorname{tr}_H^G H^*(H,k).$$

Corollary 2.4 (Okuyama). *If a homogeneous system $\{\zeta_1, \ldots, \zeta_r\}$ of parameters is taken as in the theorem above, then the tensor product $L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_{r-1}}$ is $\mathcal{H}_r(G)$-projective.*

In particular, if $r = 2$, then L_{ζ_1} is $\mathcal{H}_2(G)$-projective and the element ζ_1 is regular in $H^*(G,k)$.

The following will be used to decompose a Carlson module.

Lemma 2.5. *Let G be a finite group of p-rank two. Suppose that a set $\{\rho, \sigma\}$ is a homogeneous system of parameters of $H^*(G,k)$. Then it holds that*

$$L_{\rho\sigma} \simeq L_\rho \oplus L_\sigma.$$

3. Cohomology algebra of extraspecial p-group

Let

$$P = \langle\, a, b \mid a^p = b^p = [a,b]^p = 1, [[a,b],a] = [[a,b],b] = 1 \,\rangle$$

be an extraspecial p-group of order p^3 and exponent p. In this section, following Leary [8], we state the cohomology algebra $H^*(P,k)$. Moreover we state our key fact on which our study depends.

Definition 3.1. Let

$$c = [a,b].$$

Then $Z(P) = \langle c \rangle$. For $j = 0, \ldots, p-1$, let

$$E_j = \langle ab^j, c \rangle; \; a_j = ab^j, \; b_j = b.$$

Let

$$E_\infty = \langle b, c \rangle; \; a_\infty = b, \; b_\infty = a^{-1}.$$

We put

$$\Omega = \{0, 1, \ldots, p-1, \infty\}; \; \mathcal{E} = \{\, E_j \mid j \in \Omega \,\}.$$

The set $\mathcal{E}$ is the collection of all elementary abelian subgroups of rank two. We note that $C_P(E) = E$ for E in $\mathcal{E}$.

Definition 3.2. For j in Ω, regarding $H^1(E_j, \mathbf{F}_p)$ as $\mathrm{Hom}(E_j, \mathbf{F}_p)$, let

$$\lambda_1^{(j)} = a_j^*,\ \mu_1^{(j)} = c^*$$

and let

$$\lambda_2^{(j)} = \Delta(\lambda_1^{(j)}),\ \mu_2^{(j)} = \Delta(\mu_1^{(j)}),$$

where $\Delta : H^1(E_j, \mathbf{F}_p) \to H^2(E_j, \mathbf{F}_p)$ is the Bockstein homomorphism. Then the element b_j acts on these elements as follows:

$$(\lambda_2^{(j)})^{b_j} = \lambda_2^{(j)},\ (\mu_2^{(j)})^{b_j} = -\lambda_2^{(j)} + \mu_2^{(j)}.$$

Remark 3.1. In his report Sasaki [13] the author discussed the mod p cohomology algebra $H^*(P, k)$. There he made a stupid error, namely in Definition 4.2 in [13] he defined

$$\mu_i = b_i^* \quad i \in \Omega.$$

This should be of course

$$\mu_i = c^* \quad i \in \Omega.$$

Definition 3.3. Let us fix some classes in the cohomology algebra $H^*(P, \mathbf{F}_p)$, following Leary [8]. Regarding $H^1(P, \mathbf{F}_p)$ as $\mathrm{Hom}(P, \mathbf{F}_p)$, let

$$\alpha_1 = a^*, \qquad \beta_1 = b^*;$$
$$\alpha_2 = \Delta(\alpha_1),\ \beta_2 = \Delta(\beta_1),$$

where $\Delta : H^1(P, \mathbf{F}_p) \to H^2(P, \mathbf{F}_p)$ is the Bockstein homomorphism. Let us, as in Leary [8], denote by $\langle\ ,\ ,\ \rangle$ the Massey product. Let

$$\eta_2 = \langle \alpha_1, \alpha_1, \beta_1 \rangle,\ \theta_2 = \langle \beta_1, \beta_1, \alpha_1 \rangle;$$
$$\eta_3 = \Delta(\eta_2), \qquad \theta_3 = \Delta(\theta_2),$$

where $\Delta : H^2(P, \mathbf{F}_p) \to H^3(P, \mathbf{F}_p)$ is the Bockstein homomorphism. We let

$$\chi_{2i-1} = \mathrm{tr}_{E_\infty}^P(\mu_1^{(\infty)}(\mu_2^{(\infty)})^{i-1}),\ i = 2, \ldots, p-2,$$
$$\chi_{2i} = \mathrm{tr}_{E_\infty}^P((\mu_2^{(\infty)})^{i}),\ i = 2, \ldots, p-2,$$
$$\chi_{2p-3} = \mathrm{tr}_{E_\infty}^P(\mu_1^{(\infty)}(\mu_2^{(\infty)})^{p-2}) - \alpha_2^{p-2}\alpha_1,$$
$$\chi_{2p-2} = \mathrm{tr}_{E_\infty}^P((\mu_2^{(\infty)})^{p-1}) - \alpha_2^{p-1},$$
$$\chi_{2p-1} = \mathrm{tr}_{E_\infty}^P(\mu_1^{(\infty)}(\mu_2^{(\infty)})^{p-1}) + \alpha_2^{p-2}\eta_3.$$

Finally, we let

$$\nu = z \in H^{2p}(P, \mathbf{F}_p) \text{ in Leary [8].}$$

Theorem 3.1 (Leary [8] Theorem 6). *Let p be greater than 3. Then the cohomology algebra $H^*(P, \mathbf{F}_p)$ is generated by the classes α_i, β_i, $i = 1, 2$, η_i, θ_i, $i = 2, 3$, $\chi_i, i = 7, 8, \ldots, 2p-1$, and ν subject to the following relations:*

$$\alpha_1\beta_1 = 0,\ \alpha_2\beta_1 = \beta_2\alpha_1,\ \alpha_1\eta_2 = \beta_1\theta_2 = 0,\ \alpha_1\theta_2 = \beta_1\eta_2,$$
$$\eta_2^2 = \theta_2^2 = \eta_2\theta_2 = 0,\ \alpha_1\eta_3 = \alpha_2\eta_2,\ \beta_1\theta_3 = \beta_2\theta_2,$$
$$\eta_3\beta_1 = 2\alpha_2\theta_2 + \beta_2\eta_2,\ \theta_3\alpha_1 = 2\beta_2\eta_2 + \alpha_2\theta_2,$$
$$\eta_2\eta_3 = \theta_2\theta_3 = 0,\ \theta_2\eta_3 = -\eta_2\theta_3,\ \alpha_2\theta_3 = -\beta_2\eta_3,$$
$$\alpha_2(\alpha_2\theta_2 + \beta_2\eta_2) = \beta_2(\alpha_2\theta_2 + \beta_2\eta_2) = 0,$$

$$\alpha_2^p\beta_1-\beta_2^p\alpha_1=0,\ \alpha_2^p\beta_2-\beta_2^p\alpha_2=0,$$
$$\alpha_2^p\theta_2+\beta_2^p\eta_2=0,\ \alpha_2^p\theta_3+\beta_2^p\eta_3=0,$$
$$\chi_{2i}\alpha_1=\begin{cases}0\\ -\alpha_2^{p-1}\alpha_1\end{cases}\quad \chi_{2i}\beta_1=\begin{cases}0 & \text{for } i<p-1\\ -\beta_2^{p-1}\beta_1 & \text{for } i=p-1\end{cases},$$
$$\chi_{2i}\alpha_2=\begin{cases}0\\ -\alpha_2^{p}\end{cases}\quad \chi_{2i}\beta_2=\begin{cases}0 & \text{for } i<p-1\\ -\beta_2^{p} & \text{for } i=p-1\end{cases},$$
$$\chi_{2i}\eta_2=\begin{cases}0\\ -\alpha_2^{p-1}\eta_2\end{cases}\quad \chi_{2i}\theta_2=\begin{cases}0 & \text{for } i<p-1\\ -\beta_2^{p-1}\theta_2 & \text{for } i=p-1\end{cases},$$
$$\chi_{2i}\eta_3=\begin{cases}0\\ -\alpha_2^{p-1}\eta_3\end{cases}\quad \chi_{2i}\theta_3=\begin{cases}0 & \text{for } i<p-1\\ -\beta_2^{p-1}\theta_3 & \text{for } i=p-1\end{cases},$$
$$\chi_{2i}\chi_{2j}=\begin{cases}0 & \text{for } i+j<2p-2\\ \alpha_2^{2p-2}+\beta_2^{2p-2}-\alpha_2^{p-1}\beta_2^{p-1} & \text{for } i=j=p-1\end{cases},$$
$$\chi_{2i-1}\alpha_1=\begin{cases}0\\ -\alpha_2^{p-1}\eta_2\end{cases}\quad \chi_{2i-1}\beta_1=\begin{cases}0 & \text{for } i<p\\ \beta_2^{p-1}\theta_2 & \text{for } i=p\end{cases},$$
$$\chi_{2i-1}\alpha_2=\begin{cases}0\\ -\alpha_2^{p-1}\alpha_1\\ \alpha_2^{p-1}\eta_3\end{cases}\quad \chi_{2i-1}\beta_2=\begin{cases}0 & \text{for } i<p-1\\ -\beta_2^{p-1}\beta_1 & \text{for } i=p-1\\ -\beta_2^{p-1}\theta_3 & \text{for } i=p\end{cases},$$
$$\chi_{2i-1}\eta_2=0,\ \chi_{2i-1}\theta_2=0,$$
$$\chi_{2i-1}\eta_3=\begin{cases}0\\ -\alpha_2^{p-1}\eta_2\end{cases}\quad \chi_{2i-1}\theta_3=\begin{cases}0 & \text{for } i\neq p-1\\ -\beta_2^{p-1}\theta_2 & \text{for } i=p-1\end{cases},$$
$$\chi_{2i-1}\chi_{2j-1}=\begin{cases}0 & \text{for } i<p-1 \text{ or } j<p-1\\ \alpha_2^{2p-3}\eta_2-\beta_2^{2p-3}\theta_2+\alpha_2^{p-1}\beta_2^{p-2}\theta_2 & \text{for } i=p \text{ and } j=p-1\end{cases},$$
$$\chi_{2i-1}\chi_{2j}=\begin{cases}0 & \text{for } i<p-1 \text{ or } j<p-1\\ \alpha_2^{2p-3}\alpha_1+\beta_2^{2p-3}\beta_1-\alpha_2^{p-1}\beta_2^{p-2}\beta_1 & \text{for } i=j=p-1\\ -\alpha_2^{2p-3}\eta_3+\beta_2^{2p-3}\theta_3-\alpha_2^{p-1}\beta_2^{p-2}\theta_3 & \text{for } i=p \text{ and } j=p-1\end{cases}.$$

The following is the key fact for our investigation.

Lemma 3.2. *One has*

$$\chi_{2p-2}=\sum_{j\in\Omega}\mathrm{tr}_{E_j}^{P}((\mu_2^{(j)})^{p-1}).$$

Though the following will not be used later, it would be worthy to be noticed.

Lemma 3.3. *One has*

$$\nu=\mathrm{norm}_{E_\infty}^{P}(\mu_2^{(\infty)})\in H^{2p}(P,\mathbf{F}_p).$$

4. Finite groups with extraspecial Sylow p-subgroups

Henceforth we let k be a field of characteristic p containing $\mathbf{F}_{p^2}$. We let G denote a finite group with P as a Sylow p-subgroup, unless otherwise stated. We shall often represent by E a subgroup E_j in $\mathcal{E}$. In this case we shall write λ_2 and μ_2 for $\lambda_2^{(j)}$ and $\mu_2^{(j)}$, respectively.

Definition 4.1. We let

$$\rho = \nu^{p-1} - \chi_{2p-2}{}^p \in H^{2p(p-1)}(P,k), \ \sigma = \nu^{p-1}\chi_{2p-2} \in H^{2(p^2-1)}(P,k).$$

Note that

$$\sigma \in \sum_{E \in \mathcal{E}} \operatorname{tr}_E^P H^{2(p^2-1)}(E,k).$$

As in Tezuka-Yagita [14], we have, using Lemma 4.2, which we also need to investigate direct sum decomposition of the Carlson module L_ρ, the following.

Theorem 4.1. *The cohomologies ρ and σ are universally stable.*

Lemma 4.2. *For E in $\mathcal{E}$ one has*

(1)

$$\operatorname{res}_E \rho = \prod_{\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p} (\mu_2 - \xi\lambda_2);$$

(2)

$$\operatorname{res}_E \sigma = -\left(\lambda_2 \prod_{j \in \mathbf{F}_p} (\mu_2 - j\lambda_2)\right)^{p-1}.$$

For E_j in $\mathcal{E}$ the factor group $P/E_j = \langle \overline{b_j} \rangle$, where $\overline{b_j} = E_j b_j$, acts by conjugation on the set

$$\{ L_{\mu_2 - \xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}.$$

Since

$$L_{\mu_2-\xi\lambda_2}{}^{b_j} = L_{\mu_2-(\xi+1)\lambda_2},$$

this action induces the action of $P/E_j = \langle \overline{b_j} \rangle$ on the set $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ such that $\xi^{b_j} = 1 + \xi$ for ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Thus, if we write $(\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P$ for a quotient set of $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ under this action, then the set

$$\{ L_{\mu_2-\xi\lambda_2} \mid \xi \in (\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P \}$$

is a complete set of representatives of the conjugation on $\{ L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$.

Using Lemma 4.2, Corollary 2.4, and Lemma 2.5, we can show the following.

Theorem 4.3. (1) *The set $\{ \rho, \sigma \}$ is a system of parameters of the cohomology algebra $H^*(P,k)$.*

(2) *The element ρ is regular in $H^*(P,k)$.*

(3) *The Carlson module L_ρ is $\mathcal{E}$-projective. In fact,*

$$L_\rho = \bigoplus_{E \in \mathcal{E}} \bigoplus_{\xi \in (\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P} L_{\mu_2-\xi\lambda_2}{}^P.$$

Definition 4.2. By Theorem 4.1 we can take a class $\widetilde{\rho}$ in $H^{2p(p-1)}(G,k)$ such that

$$\mathrm{res}_P(\widetilde{\rho}) = \rho;$$

and a class $\widetilde{\sigma}$ in $H^{2(p^2-1)}(G,k)$ such that

$$\mathrm{res}_P(\widetilde{\sigma}) = \sigma.$$

Note that

(1) the set $\{\widetilde{\rho}, \widetilde{\sigma}\}$ is a system of parameters of the cohomology algebra $H^*(G,k)$;
(2) $\widetilde{\sigma} \in \sum_{E\in\mathcal{E}} \mathrm{tr}_E^G H^{2(p^2-1)}(E,k)$;
(3) the class $\widetilde{\rho}$ is regular in $H^*(G,k)$.

Since the class $\widetilde{\rho}$ is regular in $H^*(G,k)$, we obtain from the long cohomology exact sequence that

$$\dim H^{n+2p(p-1)}(G,k)/H^n(G,k)\widetilde{\rho} = \dim \mathrm{Ext}^n_{kG}(L_{\widetilde{\rho}}, k);$$
$$\dim H^{2p(p-1)-1}(G,k) = \dim(\Omega^{-1}(L_{\widetilde{\rho}}), k)_{kG}.$$

Therefore it would be useful to examine the Carlson module $L_{\widetilde{\rho}}$.

Definition 4.3. The Carlson module $L_{\widetilde{\rho}}$ is projective relative to the family $\mathcal{H}_2(G) = \{ C_G(E) \mid E \in \mathcal{E} \}$ because of Corollary 2.4. Since the subgroup E is a Sylow p-subgroup of the centralizer $C_G(E)$, the module $L_{\widetilde{\rho}}$ is $\mathcal{E}$-projective. Theorem 4.3 implies that every indecomposable direct summand has vertex some E in $\mathcal{E}$ and a source some $L_{\mu_2-\xi\lambda_2}$, $\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p$. For E in $\mathcal{E}/G$ we denote by

$$\{ X_i^{(E)} \mid i \in I^{(E)} \}$$

the set of indecomposable direct summands of the Carlson module $L_{\widetilde{\rho}}$ with vertices E. We denote by $X^{(E)}$ the direct sum of $X_i^{(E)}$s: $X^{(E)} = \bigoplus_{i\in I^{(E)}} X_i^{(E)}$.

Thus we have by Theorem 4.3 the following

Theorem 4.4. *The Carlson module $L_{\widetilde{\rho}}$ decomposes as follows:*

$$L_{\widetilde{\rho}} = \bigoplus_{E\in\mathcal{E}/G} \bigoplus_{i\in I^{(E)}} X_i^{(E)},$$

where $X_i^{(E)}$ is an indecomposable kG-module with vertex E and a source $L_{\mu_2-\xi_i\lambda_2}$ and if $i \neq j$, then $X_i^{(E)}$ and $X_j^{(E)}$ have different sources.

Definition 4.4. Let $Y_i^{(E)}$ be a Green correspondent of $X_i^{(E)}$ with respect to $(G, E, N_G(E))$. The module $Y_i^{(E)}$ is a direct summand of the Carlson module $L_{\rho'}$ of the restriction $\rho' = \widetilde{\rho}_{N_G(E)}$ by Theorem 2.1. Let us denote by $Y^{(E)}$ the direct sum of $Y_i^{(E)}$s: $Y^{(E)} = \bigoplus_{i\in I^{(E)}} Y_i^{(E)}$.

We can show

Proposition 4.5. *It holds that*

$$(Y^{(E)})^G = X^{(E)} \oplus (\text{projective}) .$$

Corollary 4.6. *One has*

$$\mathrm{Ext}^*_{kG}(L_{\widetilde{\rho}}, k) \simeq \bigoplus_{E\in\mathcal{E}/G} \mathrm{Ext}^*_{kN_G(E)}(Y^{(E)}, k).$$

In particular

$$\dim H^{2p(p-1)-1}(G,k) = \sum_{E\in\mathcal{E}/G} \dim(\Omega^{-1}(Y^{(E)}),k)_{N_G(E)}$$

and

$$\dim H^{n+2p(p-1)}(G,k) = \dim H^n(G,k) + \sum_{E\in\mathcal{E}/G} \dim \operatorname{Ext}^n_{kN_G(E)}(Y^{(E)},k).$$

Thus if we could know a direct summand $Y^{(E)}$ of the Carlson module $L_{\rho'}$ of the restriction $\rho' = \operatorname{res}_{N_G(E)} \widetilde{\rho}$, then we would know $X^{(E)}$.

Lemma 4.7. *Under the notation above, for each i in $I^{(E)}$ if we take $L_{\mu_2-\xi_i\lambda_2}$ as a source of the indecomposable $kN_G(E)$-module $Y_i^{(E)}$, then the set $\{ L_{\mu_2-\xi_i\lambda_2} \mid i \in I^{(E)} \}$ is a complete set of representatives of the action of the factor group $N_G(E)/C_G(E)$ on the set $\{ L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$.*

For each i in $I^{(E)}$, the module $Y_i^{(E)}$ would be investigated in the following way. In what follows we omit the super script $^{(E)}$ and the subscript i; namely, we denote by Y an indecomposable direct summand of $L_{\rho'}$ with vertex E and by $L_{\mu_2-\xi\lambda_2}$ a source of Y.

(1) First we investigate the inertia group
$$H_\xi = \{ g \in N_G(E) \mid {L_{\mu_2-\xi\lambda_2}}^g \simeq L_{\mu_2-\xi\lambda_2} \}.$$
In general the factor group $H_\xi/C_G(E)$ is cyclic of order l dividing p^2-1 (see Lemma 5.1).

(2) Let us denote by L_C the extension of $L_{\mu_2-\xi\lambda_2}$ to $C_G(E)$. The induced module ${L_C}^{H_\xi}$ has l indecomposable direct summands:
$${L_C}^{H_\xi} = \bigoplus_{j=0}^{l-1} M_j.$$
The module Y is the induced module ${M_j}^{N_G(E)}$ of some M_j.

(3) Let $\rho'' = \operatorname{res}_{H_\xi} \rho'$. The Carlson module $L_{\rho''}$ has M_j above as a direct summand.

(4) The module M_j would be determined by investigation of $H^*(H_\xi,k)$.

5. Green correspondents

Let the general linear group $\mathrm{GL}(2,\mathbf{F}_p)$ act on a group $E = \langle c,a \mid c^p = a^p = 1, ac = ca \rangle$ by

$$a^g = a^s c^t,\ c^g = a^u c^v \text{ for } g = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \in \mathrm{GL}(2,\mathbf{F}_p);$$

and let

$$N = E \rtimes \mathrm{GL}(2,\mathbf{F}_p).$$

Remark 5.1. The group N is called a "Pal group" in Tezuka-Yagita [14].

A Sylow p-subgroup of N is generated by a and a matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which we denote by b; we identify this p-group with our extraspecial p-group P; hence the group E is identified with E_0 in Section 3. Since the class ρ in $H^*(P,k)$ is universally stable, we can take homogeneous class ρ' in $H^{2p(p-1)}(N,k)$ such that

$$\operatorname{res}_P \rho' = \rho.$$

Our aim is to examine the indecomposable direct summands of the Carlson module $L_{\rho'}$ with vertex E.

Definition 5.1. Regarding $H^1(E,k)$ as $\mathrm{Hom}(E,k)$, we let

$$\lambda_1 = a^*,\ \mu_1 = c^*;$$

and let

$$\lambda_2 = \Delta(\lambda_1),\ \mu_2 = \Delta(\mu_1),$$

$\Delta : H^1(E,k) \to H^2(E,k)$ is the Bockstein map.

Definition 5.2. For an arbitrary element ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ we denote by $I(\xi)$ the inertia group in $\mathrm{GL}(2,\mathbf{F}_p)$ of the Carlson module $L_{\mu_2-\xi\lambda_2}$:

$$I(\xi) = \{\, g \in \mathrm{GL}(2,\mathbf{F}_p) \mid {L_{\mu_2-\xi\lambda_2}}^g \simeq L_{\mu_2-\xi\lambda_2} \text{ as } kE\text{-modules}\,\}.$$

Lemma 5.1. *Let $X^2 - eX + f$ be the minimal polynomial of ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Then we have*

$$I(\xi) = \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + u\begin{bmatrix} 0 & -f \\ 1 & e \end{bmatrix} \;\middle|\; (s,u) \in \mathbf{F}_p \times \mathbf{F}_p - (0,0) \right\};$$

the group $I(\xi)$ is cyclic of order p^2-1.

Corollary 5.2. *The general linear group $\mathrm{GL}(2,\mathbf{F}_p)$ acts transitively on the set $\{\, L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \,\}$.*

Corollary 5.2 together with Lemma 4.7 implies that there exits a unique indecomposable direct summand of the Carlson module $L_{\rho'}$ with vertex E, which we denote by Y. We take $L_{\mu_2-\xi_0\lambda_2}$ as a source of Y, where ξ_0 in $\mathbf{F}_{p^2}$ is a primitive (p^2-1)st root of unity. If we denote by $X^2 - e_0X + f_0$ the minimal polynomial of ξ_0, then we have by Lemma 5.1 that

$$H_{\xi_0} = \left\langle \begin{bmatrix} 0 & -f_0 \\ 1 & e_0 \end{bmatrix} \right\rangle \ltimes E.$$

Let $H_{\xi_0} = H_0$ and let

$$h_0 = \begin{bmatrix} 0 & -f_0 \\ 1 & e_0 \end{bmatrix}.$$

Since E is normal in N, the module Y is the induced module of an extension $M(\xi_0)$ of $L_{\mu_2-\xi_0\lambda_2}$ to the inertia group H_0: $Y = M(\xi_0)^N$. We have to specify the extension $M(\xi_0)$. The induced module ${L_{\mu_2-\xi_0\lambda_2}}^{H_0}$ decomposes as a direct sum of p^2-1 extensions $M_0, \ldots, M_{p^2-2}$:

$${L_{\mu_2-\xi_0\lambda_2}}^{H_0} = M_0 \oplus \cdots \oplus M_{p^2-2}.$$

The extension $M(\xi_0)$ is one of these extensions.

Let us investigate the p^2-1 extensions $M_0, \ldots, M_{p^2-2}$.

Definition 5.3. We let

$$u_1 = 1 + \sum_{i=0}^{p^2-2} \xi_0^{-i}(c^{h_0^i} - 1), \; u_p = 1 + \sum_{i=0}^{p^2-2} \xi_0^{-ip}(c^{h_0^i} - 1).$$

The elements u_1 and u_p are units in kE; and $kE = k\langle u_1, u_p \rangle$. Moreover it holds that

$$(u_1 - 1)^{h_0} = \xi_0(u_1 - 1), \; (u_p - 1)^{h_0} = \xi_0^p(u_p - 1).$$

The Carlson module $L_{\mu_2 - \xi_0\lambda_2}$ is described as follows by using these units.

Lemma 5.3. *It holds that*

$$L_{\mu_2-\xi_0\lambda_2} = \langle\, ((u_1 - 1)^{p-1}, 0), \; (u_p - 1, u_1 - 1)\,\rangle.$$

Definition 5.4. We define primitive idempotents in H_0 by

$$e_j = \frac{1}{p^2-1} \sum_{i=0}^{p^2-2} \xi_0^{-ji} h_0^i, \quad j = 0, \ldots, p^2 - 2.$$

It holds that

$$e_j h_0 = \xi_0^j e_j.$$

We also define one-dimensional kH_0-module k_j on which the group E acts trivially and the matrix h_0 acts as multiplication by ξ_0^j.

Definition 5.5. Let us define a kH_0-module M_0 by

$$M_0 = \langle\, (e_1(u_1 - 1)^{p-1}, 0), \; (e_1(u_p - 1), e_p(u_1 - 1))\,\rangle,$$

which is an extension of the module $L_{\mu_2-\xi_0\lambda_2}$ to the inertia group H_0. For $j = 1, \ldots, p^2 - 2$ we let

$$M_j = M_0 \otimes k_j.$$

These are the direct summands of ${L_{\mu_2-\xi_0\lambda_2}}^{H_0}$.

By direct calculation we obtain the following.

Lemma 5.4. *One has*

$$\begin{aligned}
\operatorname{hd} \Omega^{2n}(M_j) &= k_{(n+1)p+j} \oplus k_{(n+1)p+1+j};\\
\operatorname{soc} \Omega^{2n}(M_j) &= k_{np+1+j} \oplus k_{(n+1)p+j};\\
\operatorname{hd} \Omega^{2n+1}(M_j) &= k_{(n+1)p+1+j} \oplus k_{(n+2)p+j};\\
\operatorname{soc} \Omega^{2n+1}(M_j) &= k_{(n+1)p+j} \oplus k_{(n+1)p+1+j}.
\end{aligned}$$

In particular, each extension M_j is periodic of period $2(p^2 - 1)$.

The extension $M(\xi_0)$ we need is one of the M_js above; and at the same time it is a direct summand of the Carlson module $L_{\rho''}$ of $\rho'' = \operatorname{res}_{H_0} \rho'$. Using Lemma 2.2 and anlyzing the cohomology algebra $H^*(H_0, k)$, we can show the following

Lemma 5.5. *One has*

$$M(\xi_0) = M_{p^2-2}.$$

Consequently, we have

Proposition 5.6. *It holds that*

$$Y = M_{p^2-2}{}^N$$

and that

$$\operatorname{Ext}^n_{kN}(Y,k) = \begin{cases} k & \text{when } n \equiv 2(p-2)+1, 2(p-1), 2(p^2-2), 2(p^2-2)+1 \pmod{2(p^2-1)} \\ 0 & \text{otherwise} \end{cases}.$$

6. The cohomology algebra of the general linear group $\mathrm{GL}(3,\mathbf{F}_p)$

In this section, applying the facts we have established in the preceeding sections, we calculate the mod p cohomology algebra of the general linear group $\mathrm{GL}(3,\mathbf{F}_p)$.

Let $G = \mathrm{GL}(3,\mathbf{F}_p)$. Let

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the subgroup $P = \langle a, b \rangle$ is a Sylow p-subgroup of G, which is extraspecial of order p^3 and exponent p. Let us take

$$\{ E_0, E_1, E_\infty \}$$

as a complete set $\mathcal{E}/G$ of representatives of conjugacy classes of elementary abelian p-subgroups of G of rank two. Then the Carlson module $L_{\widetilde{\rho}}$ decomposes as follows:

$$L_{\widetilde{\rho}} = \bigoplus_{E \in \mathcal{E}/G} X^{(E)},$$

where $X^{(E)}$ is the sum of the indecomposable direct summands of $L_{\widetilde{\rho}}$ with vertex E (see Definition 4.3). To investigate each $X^{(E)}$ we have to know the normalizers $N_G(E)$. The following three lemmas follow from Corollary 5.2.

Lemma 6.1. *The factor group $N_G(E_0)/C_G(E_0)$ is isomorphic to* $\operatorname{Aut} E_0$ $(\simeq \mathrm{GL}(2,\mathbf{F}_p))$; *this factor group acts transitively on the set* $\{ L_{\mu_2^{(0)} - \xi\lambda_2^{(0)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$.

Lemma 6.2. *The factor group $N_G(E_1)/C_G(E_1)$ is isomorphic to the subgroup*

$$\left\{ \begin{bmatrix} t & u \\ 0 & t^2 \end{bmatrix} \;\middle|\; t, u \in \mathbf{F}_p, t \neq 0 \right\}$$

of the automorphism group $\operatorname{Aut} E_1$. *For an element ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ the inertia group H_ξ of the module $L_{\mu_2^{(1)} - \xi\lambda_2^{(1)}}$ is the centralizer $C_G(E_1)$; and hence the factor group $N_G(E_1)/C_G(E_1)$ acts transitively on the set* $\{ L_{\mu_2^{(1)} - \xi\lambda_2^{(1)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$.

Lemma 6.3. *The factor group $N_G(E_\infty)/C_G(E_\infty)$ is isomorphic to* $\operatorname{Aut} E_\infty$ $(\simeq \mathrm{GL}(2,\mathbf{F}_p))$; *this factor group acts transitively on the set* $\{ L_{\mu_2^{(\infty)} - \xi\lambda_2^{(\infty)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$.

For each E_j in $\mathcal{E}/G$ the factor group $N_G(E_j)/C_G(E_j)$ acts, by Lemmas 6.1, 6.2, 6.3, transitively on the set $\{ L_{\mu_2^{(j)} - \xi\lambda_2^{(j)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p \}$. Therefore, there exists only one indecomposable direct summand of $L_{\widetilde{\rho}}$ with vertex E_j by Lemma 4.7. Thus by Theorem 4.4 the Carlson module $L_{\widetilde{\rho}}$ decomposes as

$$L_{\widetilde{\rho}} = X_0 \oplus X_1 \oplus X_\infty,$$

where X_i is an indecomposable module with vertex E_i. Let Y_i be a Green correspondent of X_i with respect to $(G, E_i, N_G(E_i))$. The modules Y_0 and Y_∞ are the ones obtained in the previous section. Let us examine the module Y_1. Let $C_1 = C_G(E_1)$. The inertia group H_ξ in $N_1 = N_G(E_1)$ for an element ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ is the centralizer C_1. Hence, if we denote by L_{C_1} an extension of $L_{\mu_2^{(1)} - \xi\lambda_2^{(1)}}$ to the centralizer C_1, then we see that $Y_1 = {L_{C_1}}^{N_1}$. Therefore we have

$$\dim \mathrm{Ext}^n_{kN_1}(Y_1, k) = \dim \mathrm{Ext}^n_{kE_1}(L_{\mu_2^{(1)} - \xi\lambda_2^{(1)}}, k) = 2, \quad n \geq 0.$$

This together with Proposition 5.6 leads us to the following.

Theorem 6.4. *One has*

$$\dim \mathrm{Ext}^n_{kG}(L_{\widetilde{\rho}}, k) = \begin{cases} 4 & \text{when } n \equiv 2(p-2)+1, 2(p-1), 2(p^2-2), 2(p^2-2)+1 \pmod{2(p^2-1)} \\ 2 & \text{otherwise} \end{cases}.$$

Theorem 6.5. (1) *One has*

$$\dim H^{n+2p(p-1)}(G, k) = \dim H^n(G, k) + \begin{cases} 4 & \text{when } n \equiv 2(p-2)+1, 2(p-1), 2(p^2-2), 2(p^2-2)+1 \pmod{2(p^2-1)} \\ 2 & \text{otherwise} \end{cases}.$$

(2) *One has*

$$\dim H^{2p(p-1)-1}(G, k) = 4.$$

Let

$$r = 2p(p-1), \; s = 2(p^2-1).$$

Corollary 6.6. *Let $h_i = \dim H^i(G, k)$. Then the Poincare series of the cohomology algebra $H^*(G, k)$ is*

$$\frac{\left(\sum_{i=0}^{r-1} h_i X^i\right)(1 - X^s) + 2X^r \sum_{i=0}^{s-1} X^i + 2(X^{s-1} + X^s + X^{r+s-2} + X^{r+s-1})}{(1 - X^r)(1 - X^s)}.$$

We have to determine the dimensions of the cohomology groups of degree up to $r-1$. To do that we use Proposition 18 in D. J. Green [6] as in Tezuka-Yagita [14] and Milgram-Tezuka [9]. We can also find generators by the same method. Since the classes $\widetilde{\rho}$ in $H^r(G, k)$ and $\widetilde{\sigma}$ in $H^s(G, k)$ form a system of parameters, the cohomology algebra $H^*(G, k)$ is generated by finitely many homogeneous classes of degree up to $r + s - 2$ over the polynomial subalgebra $k[\widetilde{\rho}, \widetilde{\sigma}]$. First we find the classes that are stable under the Sylow normalizer $N_G(P)$. Then among the classes obtained above we find the classes which restrict to $N_G(E)$-invariant classes in the subgroups E in $\mathcal{E}/G$.

Definition 6.1. Let us define some cohomology classes of $H^*(G, k)$ as follows:

class	definition	degree
X	$A+B+\chi_{2(p-1)}$	$2(p-1)$
X_j, $j=2,\dots,p-2$	$\chi_{2(p-j)}\nu^{j-1}$	$2(p-1)j$
Ψ	$\alpha_1\alpha_2^{p-2}+\beta_1\beta_2^{p-2}+\chi_{2(p-2)+1}$	$2(p-2)+1$
Φ_j, $j=1,\dots,p-3$	$\chi_{2(p-j-2)+1}\nu^j$	$2(p-2)+1+2(p-1)j$
Ω	$\chi_{2(p-1)+1}\nu^{p-2}$	$2(p-2)+1+2(p-1)^2$
Σ	AN	$2(p^2-1)$
T	BN	$2(p^2-1)$
Γ_j, $j=2,\dots,p-1$	$\alpha_2^{p-j}\beta_2^{p-j}\nu^{j-1}$	$2(p-j)+2(p-1)j$
Δ_j, $j=2,\dots,p-1$	$\alpha_1\alpha_2^{p-1-j}\beta_2^{p-j}\nu^{j-1}$	$2(p-1-j)+1+2(p-1)j$
E_j, $j=1,\dots,p-2$	$\alpha_2^{p-2-j}\beta_2^{p-1-j}\eta_2\nu^{j-1}$	$2(p-2-j)+2(p-1)j$
Z_j, $j=1,\dots,p-2$	$\alpha_2^{p-2-j}\beta_2^{p-1-j}\eta_3\nu^{j-1}$	$2(p-2-j)+1+2(p-1)j$
H_2	$\alpha_2^{p-2}\eta_2\nu^{p-2}$	$2(p-2)+2(p-1)^2$
Θ_2	$-\beta_2^{p-2}\theta_2\nu^{p-2}$	$2(p-2)+2(p-1)^2$
H_3	$\alpha_2^{p-2}\eta_3\nu^{p-2}$	$2(p-2)+1+2(p-1)^2$
Θ_3	$-\beta_2^{p-2}\theta_3\nu^{p-2}$	$2(p-2)+1+2(p-1)^2$
Ξ	$\alpha_2^{p-2}\eta_3\nu^{p-2}A-\alpha_1\alpha_2^{p-2}\nu^{p-1}$	$2(p-2)+1+2(p-1)p$
Π	$-\beta_2^{p-2}\theta_3\nu^{p-2}B-\beta_1\beta_2^{p-2}\nu^{p-1}$	$2(p-2)+1+2(p-1)p$

Note that the classes $\widetilde{\rho}$, $\widetilde{\sigma}$, and the classes defined above are defined over the prime field $\mathbf{F}_p$.

We have the following theorem.

Theorem 6.7. *The cohomology algebra $H^*(\mathrm{GL}(3,\mathbf{F}_p),\mathbf{F}_p)$ is generated by the classes $\widetilde{\rho}$, $\widetilde{\sigma}$, and the classes defined in Definition 6.1.*

By the definitions of our generators and the relations in Theorem 3.1 we obtain

Theorem 6.8. *The generators above satisfy the relations in the tables below, where*

$$\widetilde{\rho}' = \widetilde{\rho} - X^p;$$

classes attached with dagger marks are of odd degrees; a blank entry in the upper right triangle means that corresponding product of generators has no relations; and entries lower than main diagonal are obtained from entries in the upper right triangle :

ζ	ζX	ζX_l	$\zeta \Psi$	$\zeta \Phi_l$	$\zeta \Omega$	$\zeta \Sigma$	ζT	$\zeta \Gamma_l$	$\zeta \Delta_l$	ζE_l	ζZ_l
X		0		0	$H_3 X$	$X^2 \widetilde{\rho}'$	$X^2 \widetilde{\rho}'$				
X_j		0	0	0	0	0	0	0	0	0	0
$\Psi^\dagger$			0	0	$-H_2 X$	$\Psi X \widetilde{\rho}'$	$\Psi X \widetilde{\rho}'$	$\Delta_l X$	0	0	$E_l X$
${\Phi_j}^\dagger$				0	0	0	0	0	0	0	0
$\Omega^\dagger$					0	$H_3 \Sigma$	$-\Theta_3 T$	$Z_{l-1} X \widetilde{\rho}'$	$E_{l-1} X \widetilde{\rho}'$	0	0
Σ							$X^2 \widetilde{\rho}'^2$	$\Gamma_l X \widetilde{\rho}'$	$\Delta_l X \widetilde{\rho}'$	$E_l X \widetilde{\rho}'$	$Z_l X \widetilde{\rho}'$
T								$\Gamma_l X \widetilde{\rho}'$	$\Delta_l X \widetilde{\rho}'$	$E_l X \widetilde{\rho}'$	$Z_l X \widetilde{\rho}'$
Γ_j								See below			
${\Delta_j}^\dagger$									0	0	$\Gamma_j E_l$
E_j										0	0
${Z_j}^\dagger$											0
H_2											
Θ_2											
${H_3}^\dagger$											
${\Theta_3}^\dagger$											
$\Xi^\dagger$											
$\Pi^\dagger$											
$\widetilde{\rho}$											
$\widetilde{\sigma}$											

ζ	$\zeta \Gamma_l$	$\zeta \Delta_l$	ζE_l	ζZ_l
Γ_j	$\begin{cases} \Gamma_{j+l-1} X^2 \\ \Gamma_{j+l-p} \widetilde{\rho}' \end{cases}$	$\begin{cases} \Delta_{j+l-1} X^2 \\ \Delta_{j+l-p} \widetilde{\rho}' \end{cases}$	$\begin{cases} E_{j+l-1} X^2 \\ E_{j+l-p} \widetilde{\rho}' \end{cases}$	$\begin{cases} Z_{j+l-1} X^2, & j+l \leq p \\ Z_{j+l-p} \widetilde{\rho}', & j+l > p \end{cases}$

ζ	ζH_2	$\zeta \Theta_2$	ζH_3	$\zeta \Theta_3$	$\zeta \Xi$	$\zeta \Pi$	$\zeta \widetilde{\rho}$	$\zeta \widetilde{\sigma}$
X		H_2X		H_3X	H_3X^2 $-\Psi X\widetilde{\rho}'$	H_3X^2 $-\Psi X\widetilde{\rho}'$		$-X^2\widetilde{\rho}'$
X_j	0	0	0	0	0	0		0
$\Psi^\dagger$	0	0	H_2X	H_2X	H_2X^2	H_2X^2		$-\Psi X\widetilde{\rho}'$
${\Phi_j}^\dagger$	0	0	0	0	0	0		0
$\Omega^\dagger$	0	0	0	0	$-H_2\Sigma$	$H_2X\widetilde{\rho}'$		$-H_3\Sigma - \Theta_3 T$ $+ H_3X\widetilde{\rho}'$
Σ		$H_2X\widetilde{\rho}'$		$H_3X\widetilde{\rho}'$		$H_3X^2\widetilde{\rho}'$ $-\Psi X\widetilde{\rho}'^2$		$-\Sigma^2$
T	$H_2X\widetilde{\rho}'$		$H_3X\widetilde{\rho}'$		$H_3X^2\widetilde{\rho}'$ $-\Psi X\widetilde{\rho}'^2$			$-T^2$
Γ_j	$E_{j-1}X\widetilde{\rho}'$	$E_{j-1}X\widetilde{\rho}'$	$Z_{j-1}X\widetilde{\rho}'$	$Z_{j-1}X\widetilde{\rho}'$	$Z_{j-1}X^2\widetilde{\rho}'$ $-\Delta_jX\widetilde{\rho}'$	$Z_{j-1}X^2\widetilde{\rho}'$ $-\Delta_jX\widetilde{\rho}'$		$-\Gamma_jX\widetilde{\rho}'$
${\Delta_j}^\dagger$	0	0	$E_{j-1}X\widetilde{\rho}'$	$E_{j-1}X\widetilde{\rho}'$	$E_{j-1}X^2\widetilde{\rho}'$	$E_{j-1}X^2\widetilde{\rho}'$		$-\Delta_jX\widetilde{\rho}'$
E_j	0	0	0	0	0	0		$-E_jX\widetilde{\rho}'$
${Z_j}^\dagger$	0	0	0	0	$E_jX\widetilde{\rho}'$	$E_jX\widetilde{\rho}'$		$-Z_jX\widetilde{\rho}'$
H_2	0	0	0	0	0	0		$-H_2\Sigma$
Θ_2		0	0	0	0	0		Θ_2T
${H_3}^\dagger$			0	0	$H_2\Sigma$	$H_2X\widetilde{\rho}'$		$-H_3\Sigma$
${\Theta_3}^\dagger$				0	$H_2\widetilde{\rho}'$	Θ_2T		Θ_3T
$\Xi^\dagger$					0	0		$\Xi\Sigma$
$\Pi^\dagger$						0		ΠT
$\widetilde{\rho}$								
$\widetilde{\sigma}$								$\Sigma^2 + T^2$ $-X^2\widetilde{\rho}'^2$

Theorem 6.9. *The generators of the cohomology algebra $H^*(\mathrm{GL}(3, \mathbf{F}_p), \mathbf{F}_p)$ in Theorem 6.7 and relations in Theorem 6.8 are fundamental defining relations.*

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